33 pts

1 a) Show $[a_-, a_+] = 1$ using $[x, p] = i\hbar$

8 pts

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x \mp ip\right)$$

$$\boxed{[a_-, a_+] = a_- a_+ - a_+ a_-}$$ 1 pt

$$= \frac{1}{2\hbar m\omega}\left[\left(m\omega x + ip\right)\left(m\omega x - ip\right) - \left(m\omega x - ip\right)\left(m\omega x + ip\right)\right]$$

$$\boxed{= \frac{1}{2\hbar m\omega}\left[(m\omega x)^2 + (m\omega i)(px - xp) + p^2 - (m\omega x)^2 - p^2 + (m\omega i)(px - xp)\right]}$$ 4 pts

$$= \frac{1}{2\hbar m\omega}\left[(2m\omega i)\left([p, x]\right)\right] = \frac{1}{2\hbar m\omega}(2m\omega i)(-i\hbar) = \frac{2m\omega\hbar}{2\hbar m\omega} = \boxed{1}$$

3 pts

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b) $\hat{N} = a_+ a_-$ $\qquad \hat{H}\Psi_n = \epsilon_n \Psi_n$

5 pts

3 pts

$$N\Psi_n = a_+ a_- \Psi_n = a_+\left(\sqrt{n}\,\Psi_{n-1}\right) = \sqrt{n}\sqrt{(n-1)+1}\;\Psi_n \Rightarrow \boxed{N\Psi_n = n\Psi_n}$$

$$H = \hbar\omega\left(a_+ a_- + \tfrac{1}{2}\right) = \hbar\omega\left(N + \tfrac{1}{2}\right)$$

2 pts

$$H\Psi_n = \hbar\omega\left(N + \tfrac{1}{2}\right)\Psi_n = \hbar\omega\left(n + \tfrac{1}{2}\right)\Psi_n = \epsilon_n\Psi_n \Rightarrow \boxed{\epsilon_n = \hbar\omega\left(n + \tfrac{1}{2}\right)}$$

c) $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_0\rangle + i|\Psi_1\rangle\right)$ $\qquad \langle x\rangle = ? \quad \langle p\rangle = ?$

12 pts

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x \mp ip\right) \Rightarrow \boxed{\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a_-\right) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}}\left(a_+ - a_-\right)\end{aligned}}$$

4 pts

$$\langle x\rangle = \langle\Psi|x|\Psi\rangle = \boxed{\left[\frac{1}{\sqrt{2}}\left(\langle\Psi_0| - i\langle\Psi_1|\right)\right]\left(\sqrt{\frac{\hbar}{2m\omega}}\right)\left(a_+ + a_-\right)\left[\frac{1}{\sqrt{2}}\left(|\Psi_0\rangle + i|\Psi_1\rangle\right)\right]}$$

2 pts

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle\Psi_0| - i\langle\Psi_1|\right)\left(|\Psi_1\rangle + i\sqrt{2}|\Psi_2\rangle + i|\Psi_0\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(i - i\right) \Rightarrow \boxed{\langle x\rangle = 0}$$

2 pts

2 pts

$$\langle p\rangle = \langle\Psi|p|\Psi\rangle = \boxed{\frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}\left(\langle\Psi_0| - i\langle\Psi_1|\right)\left(a_+ - a_-\right)\left(|\Psi_0\rangle + i|\Psi_1\rangle\right)}$$

$$= \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}\left(\langle\Psi_0| - i\langle\Psi_1|\right)\left(|\Psi_1\rangle + i\sqrt{2}|\Psi_2\rangle - i|\Psi_0\rangle\right)$$

2 pts

$$= \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}\left(-i - i\right) = \frac{(i)(-2i)}{2}\sqrt{\frac{\hbar m\omega}{2}} \Rightarrow \boxed{\langle p\rangle = \sqrt{\frac{\hbar m\omega}{2}}}$$

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d) Two electrons in state $\Psi_0 \Rightarrow \boxed{\Psi_T = \left(\psi_0^a\psi_0^b\right)\chi_{singlet}}$ 3 pts

8 pts $\chi_{singlet} = \boxed{\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}$ 1 pt $\Rightarrow \Psi_T$ is antisymmetric

One e^- in Ψ_0 and the other in Ψ_1:

$$\boxed{\begin{aligned}\Psi_T &= \tfrac{1}{\sqrt{2}}\left(\psi_0^a\psi_1^b - \psi_1^a\psi_0^b\right)\chi_{Triplet} \\ &\text{or} \\ \Psi_T &= \tfrac{1}{\sqrt{2}}\left(\psi_0^a\psi_1^b + \psi_1^a\psi_0^b\right)\chi_{singlet}\end{aligned}}$$

3 pts

$$\boxed{\chi_{Triplet} = \begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |\downarrow\downarrow\rangle \end{cases}}$$

1 pt

2 33 pts $|\chi\rangle = A\left(2i|\uparrow\rangle + 3|\downarrow\rangle\right)$

3 pts a) $\langle\chi|\chi\rangle = 1 \Rightarrow |A|^2\left(-2i\langle\uparrow| + 3\langle\downarrow|\right)\left(2i|\uparrow\rangle + 3|\downarrow\rangle\right) = 1$

$\Rightarrow |A|^2(4+9) = 13|A|^2 = 1 \Rightarrow \boxed{A = 1/\sqrt{13}}$

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b) Possible outcomes $\boxed{+\frac{\hbar}{2} \text{ and } -\frac{\hbar}{2}.}$ 2.5 pts

5 pts Probabilities $\boxed{4/13 \text{ and } 9/13.}$ 2.5 pts

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c) $\hat{S}_z|\chi\rangle = \frac{1}{\sqrt{13}}\left(2i\,\hat{S}_z|\uparrow\rangle + 3\,\hat{S}_z|\downarrow\rangle\right) = \frac{1}{\sqrt{13}}\left(2i\left(\frac{\hbar}{2}\right)|\uparrow\rangle + 3\left(-\frac{\hbar}{2}\right)|\downarrow\rangle\right)$

10 pts

$= \boxed{\left(\frac{\hbar}{2}\right)\left(\frac{1}{\sqrt{13}}\right)\left(2i|\uparrow\rangle - 3|\downarrow\rangle\right) \neq c|\chi\rangle \quad \text{Not eigenstate}}$ 2 pts

$\hat{S}_x|\uparrow\rangle = \left(\frac{\hbar}{2}\right)|\downarrow\rangle \rightarrow \left(\frac{\hbar}{2}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$

$\hat{S}_x|\downarrow\rangle = \left(\frac{\hbar}{2}\right)|\uparrow\rangle \Rightarrow \left(\frac{\hbar}{2}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$

so:

$\hat{S}_x|\chi\rangle = \frac{1}{\sqrt{13}}\left(2i\left(\frac{\hbar}{2}\right)|\downarrow\rangle + 3\left(\frac{\hbar}{2}\right)|\uparrow\rangle\right) = \boxed{\left(\frac{\hbar}{2}\right)\frac{1}{\sqrt{13}}\left(2i|\downarrow\rangle + 3|\uparrow\rangle\right) \neq c|\chi\rangle \quad \text{Not eigenstate}}$ 2 pts

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$\langle S_z\rangle = \langle\chi|S_z|\chi\rangle = \frac{\hbar}{2}\frac{1}{13}\left(-2i\langle\uparrow| + 3\langle\downarrow|\right)\left(2i|\uparrow\rangle - 3|\downarrow\rangle\right) = \left(\frac{\hbar}{2}\right)\left(\frac{1}{13}\right)(4-9)$

$\boxed{\langle S_z\rangle = -\frac{5}{13}\frac{\hbar}{2}}$ 3 pts

$\langle S_x\rangle = \left(\frac{\hbar}{2}\right)\frac{1}{13}\left(-2i\langle\uparrow| + 3\langle\downarrow|\right)\left(2i|\downarrow\rangle + 3|\uparrow\rangle\right) = \left(\frac{\hbar}{2}\right)\left(\frac{1}{13}\right)(-6i + 6i)$

$\boxed{\langle S_x\rangle = 0}$ 3 pts

d) $\hat{S}_x \rightarrow \left(\frac{\hbar}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\boxed{\hat{S}_x |\pm\rangle_x = \pm\left(\frac{\hbar}{2}\right)|\pm\rangle_x}$ 2 pts

8 pts

2 pts

$$\boxed{\left(\frac{\hbar}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\left(\frac{\hbar}{2}\right)\begin{pmatrix} a \\ b \end{pmatrix}} \Rightarrow a = \mp b \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalized: $|\uparrow\rangle_x \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow |\uparrow\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$

4 pts

$|\downarrow\rangle_x \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Rightarrow |\downarrow\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$

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e) $|\chi\rangle = \frac{1}{\sqrt{13}}\left(2i|\uparrow\rangle + 3|\downarrow\rangle\right)$

7 pts

$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right)$

$|\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x - |\downarrow\rangle_x\right)$

$$|\chi\rangle = \frac{1}{\sqrt{13}}\left[\frac{2i}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right) + \frac{3}{\sqrt{2}}\left(|\uparrow\rangle_x - |\downarrow\rangle_x\right)\right]$$

$$\boxed{= \frac{1}{\sqrt{26}}\left[(2i+3)|\uparrow\rangle_x + (2i-3)|\downarrow\rangle_x\right]}$$ 2 pts

Possible outcomes: $\boxed{+\frac{\hbar}{2} \text{ and } -\frac{\hbar}{2}}$ 2 pts

with probabilities: $\left|\frac{2i+3}{\sqrt{26}}\right|^2$ and $\left|\frac{2i-3}{\sqrt{26}}\right|^2$ $\Rightarrow$ $\boxed{50/50 \quad \frac{1}{2} \text{ and } \frac{1}{2}}$

2 pts

$$\frac{(-2i+3)(2i+3)}{26} = \frac{4+9}{26} = \frac{13}{26} = \frac{1}{2}$$

$\boxed{\text{So } \langle S_x\rangle = 0}$

1 pt

$$\frac{(-2i-3)(2i-3)}{26} = \frac{4+9}{26} = \frac{13}{26} = \frac{1}{2}$$

34 pts

3) a) $V = -\alpha\,\delta(x)$

9 pts

Outside the δ-well, the S.E. is: (for $x<0$):

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi} \Rightarrow \frac{d^2\psi}{dx^2} = k^2\psi \quad \text{with } k \equiv \frac{\sqrt{-2mE}}{\hbar}$$

1 pt

Since $E<0 \Rightarrow k \in \mathbb{R}$

General solution:

$$\boxed{\psi(x) = A e^{-kx} + B e^{kx} \quad (x<0)}$$

1 pt

For $x \to -\infty \Rightarrow \psi \to 0 \Rightarrow A = 0$ so

Similarly, to the right of the well:

$$\boxed{\begin{array}{ll} \psi = B e^{kx} & (x<0) \\ \psi = C e^{-kx} & (x>0) \end{array}}$$

2 pts

Continuity of $\psi \Rightarrow B = C$.

So in general: $\psi(x) = B e^{-k|x|}$

Normalizing: $\int |\psi|^2 dx = 1 = |B|^2 \left[\int_{-\infty}^{0} e^{2kx} dx + \int_{0}^{\infty} e^{-2kx} dx \right]$

$$= |B|^2 \left[\frac{1}{2k} + \frac{1}{2k} \right] = \frac{|B|^2}{k} = 1 \Rightarrow \boxed{B = \sqrt{k}}$$

3 pts

$$\boxed{\psi(x) = \sqrt{k}\, e^{-k|x|}}$$

2 pts

6) 8 pts

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\psi(x) = \epsilon\,\psi(x)$$

Integrating around $\pm\varepsilon$:

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx - \alpha\int_{-\varepsilon}^{\varepsilon}\delta\psi\,dx = \epsilon\int_{-\varepsilon}^{\varepsilon}\psi\,dx$$

2 pts

$$-\frac{\hbar^2}{2m}\left[\left.\frac{d\psi}{dx}\right|_{x=\varepsilon} - \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon}\right] - \alpha\psi(0) = 0$$

$$-\frac{\hbar^2}{2m}\Delta\left(\frac{d\psi}{dx}\right) = \alpha\psi(x=0) \Rightarrow$$

2 pts

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(x=0)$$

$$\frac{d\psi}{dx} = -\sqrt{k}\,k e^{-kx} \quad (x>0) \Rightarrow \left.\frac{d\psi}{dx}\right|_{x=+\varepsilon} = -\sqrt{k}\,k \quad \left(\begin{array}{l}\text{for } \varepsilon\to 0\\ \text{at } x>0\end{array}\right)$$

2 pts

$$\frac{d\psi}{dx} = \sqrt{k}\,k e^{kx} \quad (x<0) \Rightarrow \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon} = \sqrt{k}\,k \quad \left(\begin{array}{l}\text{for } \varepsilon\to 0\\ \text{at } x<0\end{array}\right)$$

$$\Delta\left(\frac{d\psi}{dx}\right) = -2k\sqrt{k} = \frac{2m\alpha}{\hbar^2}\psi(x=0) = -\frac{2m\alpha}{\hbar^2}\sqrt{k} \Rightarrow -2k = \frac{\alpha 2m}{\hbar^2}$$

$$k = \frac{\sqrt{-2m\epsilon}}{\hbar} \Rightarrow \frac{\sqrt{-2m\epsilon}}{\hbar} = -\frac{2\alpha m}{2\hbar^2} \Rightarrow \boxed{\epsilon = -\frac{m\alpha^2}{2\hbar^2}}$$

2 pts

9 pts

c) $|\alpha(t)\rangle$ $\qquad$ $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}$

4 pts

$$\Phi(k,t) = \langle k|\alpha(t)\rangle = \boxed{\langle k| \left(\int |x\rangle\langle x| dx \right) |\alpha(t)\rangle}$$

$$= \int \langle k|x\rangle \langle x|\alpha(t)\rangle dx \Rightarrow \boxed{\Phi(k,t) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x,t)\, dx}$$

$$\boxed{\langle k|x\rangle = (\langle x|k\rangle)^*}$$

3 pts

2 pts

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8 pts

d) $\Psi(x,t=0) = A e^{-\kappa|x|}$

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k,t=0)\, e^{i(kx-\omega t)} dk}$$

1 pt

1 pt

$$\boxed{\phi(k,t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t=0)\, e^{-ikx} dx} = \boxed{\frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\kappa|x|} e^{-ikx} dx}$$

2 pt

$$= \frac{A}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(\kappa-ik)x} dx + \int_{0}^{\infty} e^{-(\kappa+ik)x} dx \right] = \frac{A}{\sqrt{2\pi}} \left[\frac{1}{(\kappa-ik)}(1-0) + \frac{1}{(\kappa+ik)}(0-\ldots \right.$$

$$\boxed{= \frac{A}{\sqrt{2\pi}} \frac{2\kappa}{\kappa^2+k^2} = \phi(k)}$$

4 pts

or with $A = \sqrt{\kappa}$ $\Rightarrow$ $\boxed{\phi(k) = \frac{1}{\sqrt{2\pi}} \frac{\kappa\sqrt{\kappa}}{\kappa^2+k^2}}$